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Compound Determinants.

BY WILLIAM H. METZLER, PH. D.

It is proposed in this paper to show, by a method similar to that employed in the Am. Jour. of Math., vol. XVI, No. 3, pp. 131–150, how the value of certain minors* of the m^{th} compound of a given determinant may very easily be found in terms of the given determinant and its minors.

1. If Δ denote a determinant of order n , then $\Delta(m)$ will denote the m^{th} compound of Δ , and $\Delta(n-m)$ will be termed the adjugate of $\Delta(m)$. We have the well-known relation connecting minors of $\Delta(m)$ with those of $\Delta(n-m)$, viz. *any minor of $\Delta(m)$ of order k is equal to the complementary of the corresponding minor of $\Delta(n-m)$ multiplied by $\Delta^{k-(n-1)m}$.*†

2. For the sake of definiteness, let us start with a determinant of order five :

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

then

$$\Delta_{(2)} = \begin{vmatrix} A_{123}^{123} & A_{123}^{124} & A_{123}^{125} & A_{123}^{134} & A_{123}^{135} & A_{123}^{145} & A_{123}^{234} & A_{123}^{235} & A_{123}^{245} & A_{123}^{345} \\ A_{124}^{123} & A_{124}^{124} & A_{124}^{125} & A_{124}^{134} & A_{124}^{135} & A_{124}^{145} & A_{124}^{234} & A_{124}^{235} & A_{124}^{245} & A_{124}^{345} \\ A_{125}^{123} & A_{125}^{124} & A_{125}^{125} & A_{125}^{134} & A_{125}^{135} & A_{125}^{145} & A_{125}^{234} & A_{125}^{235} & A_{125}^{245} & A_{125}^{345} \\ A_{134}^{123} & A_{134}^{124} & A_{134}^{125} & A_{134}^{134} & A_{134}^{135} & A_{134}^{145} & A_{134}^{234} & A_{134}^{235} & A_{134}^{245} & A_{134}^{345} \\ A_{135}^{123} & A_{135}^{124} & A_{135}^{125} & A_{135}^{134} & A_{135}^{135} & A_{135}^{145} & A_{135}^{234} & A_{135}^{235} & A_{135}^{245} & A_{135}^{345} \\ A_{145}^{123} & A_{145}^{124} & A_{145}^{125} & A_{145}^{134} & A_{145}^{135} & A_{145}^{145} & A_{145}^{234} & A_{145}^{235} & A_{145}^{245} & A_{145}^{345} \\ A_{234}^{123} & A_{234}^{124} & A_{234}^{125} & A_{234}^{134} & A_{234}^{135} & A_{234}^{145} & A_{234}^{234} & A_{234}^{235} & A_{234}^{245} & A_{234}^{345} \\ A_{235}^{123} & A_{235}^{124} & A_{235}^{125} & A_{235}^{134} & A_{235}^{135} & A_{235}^{145} & A_{235}^{234} & A_{235}^{235} & A_{235}^{245} & A_{235}^{345} \\ A_{245}^{123} & A_{245}^{124} & A_{245}^{125} & A_{245}^{134} & A_{245}^{135} & A_{245}^{145} & A_{245}^{234} & A_{245}^{235} & A_{245}^{245} & A_{245}^{345} \\ A_{345}^{123} & A_{345}^{124} & A_{345}^{125} & A_{345}^{134} & A_{345}^{135} & A_{345}^{145} & A_{345}^{234} & A_{345}^{235} & A_{345}^{245} & A_{345}^{345} \end{vmatrix},$$

* All those which are expressible as a product of minors of the given determinant.

† Vide Muir's "Theory of Determinants," §175.

$$\Delta_{(3)} = \begin{vmatrix} A_{12}_{12} & A_{12}_{13} & A_{12}_{14} & A_{12}_{15} & A_{12}_{23} & A_{12}_{24} & A_{12}_{25} & A_{12}_{34} & A_{12}_{35} & A_{12}_{45} \\ A_{13}_{12} & A_{13}_{13} & A_{13}_{14} & A_{13}_{15} & A_{13}_{23} & A_{13}_{24} & A_{13}_{25} & A_{13}_{34} & A_{13}_{35} & A_{13}_{45} \\ A_{14}_{12} & A_{14}_{13} & A_{14}_{14} & A_{14}_{15} & A_{14}_{23} & A_{14}_{24} & A_{14}_{25} & A_{14}_{34} & A_{14}_{35} & A_{14}_{45} \\ A_{15}_{12} & A_{15}_{13} & A_{15}_{14} & A_{15}_{15} & A_{15}_{23} & A_{15}_{24} & A_{15}_{25} & A_{15}_{34} & A_{15}_{35} & A_{15}_{45} \\ A_{23}_{12} & A_{23}_{13} & A_{23}_{14} & A_{23}_{15} & A_{23}_{23} & A_{23}_{24} & A_{23}_{25} & A_{23}_{34} & A_{23}_{35} & A_{23}_{45} \\ A_{24}_{12} & A_{24}_{13} & A_{24}_{14} & A_{24}_{15} & A_{24}_{23} & A_{24}_{24} & A_{24}_{25} & A_{24}_{34} & A_{24}_{35} & A_{24}_{45} \\ A_{25}_{12} & A_{25}_{13} & A_{25}_{14} & A_{25}_{15} & A_{25}_{23} & A_{25}_{24} & A_{25}_{25} & A_{25}_{34} & A_{25}_{35} & A_{25}_{45} \\ A_{34}_{12} & A_{34}_{13} & A_{34}_{14} & A_{34}_{15} & A_{34}_{23} & A_{34}_{24} & A_{34}_{25} & A_{34}_{34} & A_{34}_{35} & A_{34}_{45} \\ A_{35}_{12} & A_{35}_{13} & A_{35}_{14} & A_{35}_{15} & A_{35}_{23} & A_{35}_{24} & A_{35}_{25} & A_{35}_{34} & A_{35}_{35} & A_{35}_{45} \\ A_{45}_{12} & A_{45}_{13} & A_{45}_{14} & A_{45}_{15} & A_{45}_{23} & A_{45}_{24} & A_{45}_{25} & A_{45}_{34} & A_{45}_{35} & A_{45}_{45} \end{vmatrix}$$

and

$$\Delta_{(4)} = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{vmatrix}$$

By Laplace's theorem we have

$$\left. \begin{aligned} A_{55} &= a_{11}A_{15}_{15} - a_{12}A_{15}_{25} + a_{13}A_{15}_{35} - a_{14}A_{15}_{45}, \\ &= -a_{21}A_{25}_{15} + a_{22}A_{25}_{25} - a_{23}A_{25}_{35} + a_{24}A_{25}_{45} = \text{etc.}, \\ &= a_{11}A_{15}_{15} - a_{21}A_{25}_{15} + a_{31}A_{35}_{15} - a_{41}A_{45}_{15} = \text{etc.}, \\ A_{24} &= a_{11}A_{12}_{14} - a_{12}A_{12}_{24} + a_{13}A_{12}_{34} - a_{15}A_{12}_{45} = \text{etc.} \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} 0 &= -a_{21}A_{15}_{15} + a_{22}A_{25}_{25} - a_{23}A_{15}_{35} + a_{24}A_{15}_{45}, \\ &= a_{31}A_{15}_{15} - a_{32}A_{25}_{25} + a_{33}A_{15}_{35} - a_{34}A_{15}_{45} = \text{etc.}, \\ &= -a_{12}A_{15}_{15} + a_{22}A_{25}_{15} - a_{32}A_{35}_{15} + a_{42}A_{45}_{15} = \text{etc.} \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} A_{45}_{45} &= a_{33}A_{345}_{345} - a_{32}A_{345}_{245} + a_{31}A_{345}_{145} = \text{etc.}, \\ &= a_{33}A_{345}_{345} - a_{23}A_{245}_{345} + a_{13}A_{145}_{345} = \text{etc.}, \\ A_{24}_{35} &= a_{11}A_{124}_{135} - a_{12}A_{124}_{235} + a_{14}A_{124}_{345} = \text{etc.} \end{aligned} \right\} \quad (3)$$

$$\begin{aligned} 0 &= -a_{23}A_{345} + a_{22}A_{345} - a_{21}A_{345} = \text{etc.}, \\ &= a_{51}A_{124} - a_{52}A_{124} + a_{51}A_{124} = \text{etc.} \end{aligned} \quad (4)$$

$$\begin{aligned} A_{55} &= A_{345}A_{125} - A_{345}A_{125} + A_{345}A_{125} \\ &\quad + A_{345}A_{125} - A_{345}A_{125} + A_{345}A_{125} = \text{etc.}, \\ A_{32} &= A_{123}A_{345} - A_{123}A_{345} + A_{123}A_{345} \\ &\quad + A_{123}A_{345} - A_{123}A_{345} + A_{123}A_{345} = \text{etc.} \end{aligned} \quad (5)$$

$$\begin{aligned} 0 &= A_{345}A_{135} - A_{345}A_{135} + A_{345}A_{135} \\ &\quad + A_{345}A_{135} - A_{345}A_{135} + A_{345}A_{135} = \text{etc.} \end{aligned} \quad (6)$$

Operating on these equations by the *Law of Complementaries** we get :

$$\begin{aligned} \Delta a_{55} &= A_{11}A_{234} - A_{12}A_{234} + A_{13}A_{234} - A_{14}A_{234}, \\ &= -A_{21}A_{134} + A_{22}A_{134} - A_{23}A_{134} + A_{24}A_{134} = \text{etc.}, \\ &= A_{11}A_{234} - A_{21}A_{134} + A_{31}A_{124} - A_{41}A_{123} = \text{etc.}, \\ \Delta a_{24} &= A_{11}A_{345} - A_{12}A_{345} + A_{13}A_{345} - A_{15}A_{345} = \text{etc.} \end{aligned} \quad (1')$$

$$\begin{aligned} 0 &= -A_{21}A_{234} + A_{22}A_{234} - A_{23}A_{234} + A_{24}A_{234} = \text{etc.}, \\ &= -A_{12}A_{234} + A_{22}A_{134} - A_{32}A_{124} + A_{42}A_{123} = \text{etc.} \end{aligned} \quad (2')$$

$$\begin{aligned} \Delta A_{123} &= A_{33}A_{12} - A_{32}A_{12} + A_{31}A_{12} = \text{etc.} \\ &= A_{33}A_{12} - A_{23}A_{13} + A_{13}A_{23} = \text{etc.}, \\ \Delta A_{135} &= A_{11}A_{35} - A_{12}A_{35} + A_{14}A_{35} = \text{etc.} \end{aligned} \quad (3')$$

$$\begin{aligned} 0 &= -A_{23}A_{12} + A_{22}A_{12} - A_{21}A_{12} = \text{etc.} \\ &= A_{51}A_{35} - A_{52}A_{35} + A_{54}A_{35} = \text{etc.} \end{aligned} \quad (4')$$

* Vide Muir, "Theory of Determinants," §98.

$$\left. \begin{aligned} \Delta A_{55} &= A_{12} A_{34} - A_{13} A_{24} + A_{14} A_{23} + A_{12} A_{34} \\ &\quad - A_{12} A_{34} + A_{12} A_{34} = \text{etc.}, \\ \Delta a_{32} &= A_{45} A_{12} - A_{45} A_{12} + A_{45} A_{12} + A_{45} A_{12} \\ &\quad - A_{45} A_{12} + A_{45} A_{12} = \text{etc.} \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} 0 &= A_{12} A_{24} - A_{13} A_{24} + A_{14} A_{23} + A_{12} A_{24} \\ &\quad - A_{12} A_{24} + A_{12} A_{24} = \text{etc.} \end{aligned} \right\} \quad (6')$$

Operating on equations (3) and (4) by the *Law of Extensible Minors*,* we get

$$\left. \begin{aligned} \alpha_{55} A_{44} &= A_{34} A_{124} - A_{34} A_{124} + A_{34} A_{124} = \text{etc.}, \\ &= A_{34} A_{124} - A_{24} A_{134} + A_{14} A_{234} = \text{etc.}, \\ \alpha_{45} A_{23} &= A_{12} A_{235} - A_{12} A_{235} + A_{12} A_{235} = \text{etc.} \end{aligned} \right\} \quad (3'')$$

$$\left. \begin{aligned} 0 &= -A_{34} A_{134} + A_{34} A_{134} - A_{34} A_{134} = \text{etc.}, \\ &= A_{12} A_{123} - A_{12} A_{123} + A_{12} A_{123} = \text{etc.} \end{aligned} \right\} \quad (4'')$$

3. If for the sake of uniformity we write A for Δ , A_r for Ars , $A_{23 \dots n}$ for a_{11} , etc., then in each factor of every term of the foregoing expressions there are two lines of suffixes. I shall refer to them as the upper and lower. We may make the equations homogeneous in the A 's by multiplying any term when necessary by $A_{123 \dots n}$ which is unity.

4. If we are given any combination of n numbers k at a time, the combination of the remaining $(n-k)$ numbers is said to be the complementary with respect to n of the given combination. Considering any n numbers $a_1, a_2, a_3, \dots, a_n$, let $a_{m+1}, a_{m+2}, \dots, a_n$ denote the combination complementary with respect to n of the combination $a_1, a_2, \dots, a_m$.

* Muir, "Theory of Determinants," §179.

6. Since the numbers in the upper line denote the rows and the numbers in the lower line denote the columns, it is evident that putting an α equal to a γ is equivalent to making two rows identical, and putting a β equal to a δ is equivalent to making two columns identical. If an α be put equal to a γ or a β to a δ , then two α 's or two β 's become identical and the expression vanishes, and the equation is of the form of (2), (4), (6), (2'), (4'), (6'), (4''), (6'').

7. Operating by the *Law of Complementaries* is equivalent to replacing each line of every suffix by its complementary with respect to n , and operating by the *Law of Extensible Minors* is equivalent to striking out the same j c 's and the same j d 's from the suffix of each factor of every term. It is possible therefore to reduce all equations to the form of equation (B). It may often happen that the same result is obtained by the operation of these two laws.

8. Let $(n|m)_1, (n|m)_2, \dots, (n|m)_\mu$ represent the $n_m = \mu$ combinations of the n numbers 1, 2, 3, $\dots$, n taken m at a time,* and let $(\bar{n}|m)_1, (\bar{n}|m)_2, \dots, (\bar{n}|m)_\mu$ denote the complementary combinations. If we take any combination of the numbers m at a time and combine the numbers in it in all possible ways l at a time, there would be $m_l = \lambda$ such combinations. Let

$$(n|m|l)_1, (n|m|l)_2, \dots, (n|m|l)_\lambda$$

denote the λ combinations of the numbers in the combination $(n|m)_\alpha$ taken l at a time, and let $(n|\bar{m}|l)_\beta$ denote the combination complementary with respect to m of the combination $(n|m|l)_\alpha$, that is, the combination formed by the numbers remaining after the numbers in the combination $(n|m|l)_\alpha$ are taken from the combination $(n|m)_\alpha$. Let $(n|m|l)_\alpha (\bar{n}|\bar{m}|l)_\beta$ denote the combination of the numbers in the two combinations $(n|m|l)_\alpha$ and $(\bar{n}|\bar{m}|l)_\beta$, and so in general one combination following another will denote the combination of the numbers in the two combinations.

* Let it always be understood that the numbers in every combination are arranged in their natural order.

9. With this notation, equation (B) may be written as follows:

$$\sum_{\substack{\beta \text{ or } \delta = \lambda \\ \beta \text{ or } \delta = 1}} (-1)^{\nu} A_{\substack{(n | m | l) \\ \alpha \quad \beta \\ (n | m | l) \\ \gamma \quad \delta}} A_{\substack{(n | \bar{m} | l) \\ \alpha \quad \beta \\ (n | \bar{m} | l) \\ \gamma \quad \delta}} = A \cdot A_{\substack{(n | m) \\ \alpha \\ (n | m) \\ \gamma}} \quad (C)$$

where $\sum_{\substack{\beta \text{ or } \delta = \lambda \\ \beta \text{ or } \delta = 1}}$ means that when β varies from 1 to λ , δ is constant, and when δ varies from 1 to λ , β is constant. The value of ν is the same as before.

10. If we operate on equation (C) by the *Law of Complementaries* we get

$$\begin{aligned} \sum_{\substack{\beta \text{ or } \delta = \lambda \\ \beta \text{ or } \delta = 1}} (-1)^{\nu} A_{\substack{(n | \bar{m} | l) (\bar{n} | m) \\ \alpha \quad \beta \quad \alpha \\ (n | \bar{m} | l) (\bar{n} | m) \\ \gamma \quad \delta \quad \gamma}} A_{\substack{(n | m | l) (\bar{n} | m) \\ \alpha \quad \beta \quad \alpha \\ (n | m | l) (\bar{n} | m) \\ \gamma \quad \delta \quad \gamma}} \\ = A_{\substack{(\bar{n} | m) (n | m) \\ \alpha \quad \alpha \\ (\bar{n} | m) (n | m) \\ \gamma \quad \gamma}} A_{\substack{(\bar{n} | m) \\ \alpha \\ (\bar{n} | m) \\ \gamma}} \\ = A_{\substack{(\bar{n} | m) \\ \alpha \\ (\bar{n} | m) \\ \gamma}} \end{aligned}$$

since $A_{\substack{(\bar{n} | m) (n | m) \\ \alpha \quad \alpha \\ (n | m) (n | m) \\ \gamma \quad \gamma}} = 1$.

This is the equation that would be obtained on expanding the minor $A_{\substack{(\bar{n} | m) \\ \alpha \\ (\bar{n} | m) \\ \gamma}}$ by Laplace's theorem.

11. Let A denote a determinant of order n , let $A_{(m)}$ denote the m^{th} compound of A , and let $A_{\substack{(n | m) \\ \alpha \\ (n | m) \\ \gamma}}$ denote the l^{th} compound of $A_{(n | m)}$, etc.

The upper line of the suffix is the same for every constituent in the same row of $A_{(m)}$ and the lower line of the suffix is the same for every constituent in the same column.*

The upper lines of the suffixes of the constituents in the columns from top to bottom and the lower lines of the suffixes of the constituents in the rows from left to right are the combinations $(\bar{n} | m)_1, (\bar{n} | m)_2, \dots, (\bar{n} | m)_\mu$.

* If the rows and columns were interchanged, this statement would be reversed.

12. The determinant $A_{\substack{(\bar{n} | m) \\ (\bar{n} | m) \\ \gamma}}^{\substack{a \\ a \\ \gamma}}(l)$

$$\begin{aligned} &\equiv \left(A_{\substack{(\bar{n} | m) (n | m | l) \\ (\bar{n} | m) (n | m | l) \\ \gamma \quad \gamma \quad 1}}^{\substack{a \\ a \\ \gamma}} A_{\substack{(\bar{n} | m) (n | m | l) \\ (\bar{n} | m) (n | m | l) \\ \gamma \quad \gamma \quad 2}}^{\substack{a \\ a \\ \gamma}} \cdots A_{\substack{(\bar{n} | m) (n | m | l) \\ (\bar{n} | m) (n | m | l) \\ \gamma \quad \gamma \quad \lambda}}^{\substack{a \\ a \\ \gamma}} \right) \\ &= A_{\substack{(\bar{n} | m) \\ (\bar{n} | m) \\ \gamma}}^{(m-1)m-l-1} = A_{\substack{(\bar{n} | m) \\ (\bar{n} | m) \\ \gamma}}^{(m-1)l}. \end{aligned} \quad (7)$$

Operating on this equation by the *Law of Extensible Minors* we get

$$\begin{aligned} &\left(A_{\substack{(n | m | l) \\ (n | m | l) \\ \gamma \quad 1}}^{\substack{a \\ a \\ \gamma}} A_{\substack{(n | m | l) \\ (n | m | l) \\ \gamma \quad 2}}^{\substack{a \\ a \\ \gamma}} \cdots A_{\substack{(n | m | l) \\ (n | m | l) \\ \gamma \quad \lambda}}^{\substack{a \\ a \\ \gamma}} \right) \\ &= A_{\substack{(n | m) \\ (n | m) \\ \gamma}}^{(m-1)l} \cdot A_{\substack{(n | m) \\ (n | m) \\ \gamma}}^{(m-1)l-1}. \end{aligned} \quad (8)$$

This is Sylvester's theorem,* since the constituents of the determinant on the left-hand side of the equation consist of the minor $A_{\substack{(n | m) \\ (n | m) \\ \gamma}}$ bordered in all possible ways with l of the remaining rows and columns. It is evidently a minor of order λ of the determinant $A_{(n-l)}$, the $(n-l)^{\text{th}}$ compound of A .

Similarly $A_{\substack{(\bar{n} | m) \\ (\bar{n} | m) \\ \gamma}}^{(m-l)}$

$$\begin{aligned} &\equiv \left(A_{\substack{(\bar{n} | m) (n | \bar{m} | l) \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \quad \gamma \quad 1}}^{\substack{a \\ a \\ \gamma}} A_{\substack{(\bar{n} | m) (n | \bar{m} | l) \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \quad \gamma \quad 2}}^{\substack{a \\ a \\ \gamma}} \cdots A_{\substack{(\bar{n} | m) (n | \bar{m} | l) \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \quad \gamma \quad \lambda}}^{\substack{a \\ a \\ \gamma}} \right) \\ &= A_{\substack{(\bar{n} | m) \\ (\bar{n} | m) \\ \gamma}}^{(m-1)l-1}. \end{aligned} \quad (9)$$

Operating on this by the same law we get

$$\begin{aligned} &\left(A_{\substack{(n | \bar{m} | l) \\ (n | \bar{m} | l) \\ \gamma \quad 1}}^{\substack{a \\ a \\ \gamma}} A_{\substack{(n | \bar{m} | l) \\ (n | \bar{m} | l) \\ \gamma \quad 2}}^{\substack{a \\ a \\ \gamma}} \cdots A_{\substack{(n | \bar{m} | l) \\ (n | \bar{m} | l) \\ \gamma \quad \lambda}}^{\substack{a \\ a \\ \gamma}} \right) \\ &= A_{\substack{(n | m) \\ (n | m) \\ \gamma}}^{(m-1)l-1} A_{\substack{(n | m) \\ (n | m) \\ \gamma}}^{(m-1)l}. \end{aligned} \quad (10)$$

Equation (4) might have been obtained from equation (1) by the *Law of Complementaries*.

* Vide Philosophical Magazine for 1851; also Camb. and Dublin Math. Jour., VIII, 60.

13. The k^{th} compound of $A_{\begin{smallmatrix} (n | \bar{m} | l) \\ \alpha \beta \\ (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}} \begin{smallmatrix} (\bar{n} | m) \\ \alpha \\ (\bar{n} | m) \\ \gamma \end{smallmatrix}$ is

$$\left(A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}} \begin{smallmatrix} (n | m | l | k) \\ \alpha \beta 1 \\ (n | m | l | k) \\ \gamma \delta 1 \end{smallmatrix} } A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m | l | k) \\ \alpha \beta 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \delta \sigma \end{smallmatrix}} \right)_{\sigma=l_k}$$

and is equal to

$$A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k}. \quad (11)$$

Operating on this equation by the *Law of Extensible Minors* we get

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l | k) \\ \alpha \beta 1 \\ (\bar{n} | m) (n | \bar{m} | l | k) \\ \gamma \delta 1 \end{smallmatrix}} } A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l | k) \\ \alpha \beta 2 \\ (\bar{n} | m) (n | \bar{m} | l | k) \\ \gamma \delta 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l | k) \\ \alpha \beta \sigma \\ (\bar{n} | m) (n | \bar{m} | l | k) \\ \gamma \delta \sigma \end{smallmatrix}} \right) \\ &= A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k} A_{\begin{smallmatrix} (\bar{n} | m) (n | \bar{m} | l) \\ \alpha \beta \\ (\bar{n} | m) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k-1} \end{aligned} \quad (12)$$

and

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \alpha \beta 1 \\ (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \gamma \delta 1 \end{smallmatrix}} } A_{\begin{smallmatrix} (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \alpha \beta 2 \\ (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \gamma \delta 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \alpha \beta \sigma \\ (n | \bar{m} | l) (n | \bar{m} | l | k) \\ \gamma \delta \sigma \end{smallmatrix}} \right) \\ &= A_{\begin{smallmatrix} (n | \bar{m} | l) (n | \bar{m} | l) \\ \alpha \beta \\ (n | \bar{m} | l) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k} A_{\begin{smallmatrix} (n | \bar{m} | l) (n | \bar{m} | l) \\ \alpha \beta \\ (n | \bar{m} | l) (n | \bar{m} | l) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k-1} \end{aligned} \quad (13)$$

and

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | l | k) (n | m | l | k) \\ \alpha \beta 1 \\ (n | m | l | k) (n | m | l | k) \\ \gamma \delta 1 \end{smallmatrix}} } A_{\begin{smallmatrix} (n | m | l | k) (n | m | l | k) \\ \alpha \beta 2 \\ (n | m | l | k) (n | m | l | k) \\ \gamma \delta 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | l | k) (n | m | l | k) \\ \alpha \beta \sigma \\ (n | m | l | k) (n | m | l | k) \\ \gamma \delta \sigma \end{smallmatrix}} \right) \\ &= A_{\begin{smallmatrix} (n | m | l | k) (n | m | l | k) \\ \alpha \beta \\ (n | m | l | k) (n | m | l | k) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k} A_{\begin{smallmatrix} (n | m | l | k) (n | m | l | k) \\ \alpha \beta \\ (n | m | l | k) (n | m | l | k) \\ \gamma \delta \end{smallmatrix}}^{(l-1)k-1}. \end{aligned} \quad (14)$$

14. If the constituents in the intersection of the last r columns and the last $(n-r)$ rows are zero, then $A = (-1)^{r(n-r)} A_{\begin{smallmatrix} (n | r) \\ \alpha \lambda \\ (\bar{n} | \frac{1}{n-r}) \\ \gamma \end{smallmatrix}} A_{\begin{smallmatrix} (\bar{n} | r) \\ \alpha \\ (\bar{n} | \frac{1}{n-r}) \\ \gamma \end{smallmatrix}}$ and equation (8) becomes

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | l) (n | m | l) \\ \alpha 1 \\ (n | m | l) (n | m | l) \\ \gamma 1 \end{smallmatrix}} } A_{\begin{smallmatrix} (n | m | l) (n | m | l) \\ \alpha 2 \\ (n | m | l) (n | m | l) \\ \gamma 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | l) (n | m | l) \\ \alpha \lambda \\ (n | m | l) (n | m | l) \\ \gamma \lambda \end{smallmatrix}} \right) \\ &= (-1)^{r(n-r)(m-1)l} A_{\begin{smallmatrix} (n | r) \\ \alpha \lambda \\ (\bar{n} | \frac{1}{n-r}) \\ \gamma \end{smallmatrix}}^{(m-1)l} A_{\begin{smallmatrix} (\bar{n} | r) \\ \alpha \\ (\bar{n} | \frac{1}{n-r}) \\ \gamma \end{smallmatrix}}^{(m-1)l} A_{\begin{smallmatrix} (n | m) \\ \alpha \\ (n | \frac{1}{n-r}) \\ \gamma \end{smallmatrix}}^{m-1-l}. \end{aligned} \quad (15)$$

If $n - r = r$, then equation (15) becomes

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | l) \\ \alpha & 1 \\ (n | m | l) \\ \gamma & 1 \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | l) \\ \alpha & 2 \\ (n | m | l) \\ \gamma & 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | l) \\ \alpha & \lambda \\ (n | m | l) \\ \gamma & \lambda \end{smallmatrix}} \right) \\ & = (-1)^{r^2(m-1)_l} A_{\begin{smallmatrix} (m-1)_l \\ (n | r) \\ 1 \end{smallmatrix}} A_{\begin{smallmatrix} (\bar{n} | r) \\ 1 \\ (n | n-r) \\ 1 \end{smallmatrix}} A_{\begin{smallmatrix} (m-1)_{l-1} * \\ (n | m) \\ \alpha \\ (n | m) \\ \gamma \end{smallmatrix}}. \end{aligned} \quad (16)$$

Similarly for equations (10) and (14).

15. The determinant

$$\left(A_{\begin{smallmatrix} (n | m | l | k) \\ \alpha & \beta & 1 \\ (n | m | l | k) \\ \gamma & \delta & 1 \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | l | k) \\ \alpha & \beta & 2 \\ (n | m | l | k) \\ \gamma & \delta & 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | l | k) \\ \alpha & \beta & \sigma \\ (n | m | l | k) \\ \gamma & \delta & \sigma \end{smallmatrix}} \right)$$

is a minor of order σ of

$$\left(A_{\begin{smallmatrix} (n | m | k) \\ \alpha & 1 \\ (n | m | k) \\ \gamma & 1 \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | k) \\ \alpha & 2 \\ (n | m | k) \\ \gamma & 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | k) \\ \alpha & \rho \\ (n | m | k) \\ \gamma & \rho \end{smallmatrix}} \right)^{\rho = m_k}$$

and by equation (14) is equal to

$$A^{(l-1)_k} A_{\begin{smallmatrix} (l-1)_{k-1} \dagger \\ (n | m | l) \\ \alpha & \beta \\ (n | m | l) \\ \gamma & \delta \end{smallmatrix}}.$$

16. The determinant

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | k) \\ \alpha & 1 \\ (n | m | k) \\ \gamma & 1 \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | k) \\ \alpha & 2 \\ (n | m | k) \\ \gamma & 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | k) \\ \alpha & \rho \\ (n | m | k) \\ \gamma & \rho \end{smallmatrix}} \right) \\ & = A^{(m-1)_k} A_{\begin{smallmatrix} (m-1)_{k-1} \\ (n | m) \\ \alpha \\ (n | m) \\ \gamma \end{smallmatrix}}, \end{aligned}$$

and the determinant

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | \bar{m} | k) \\ \alpha & 1 \\ (n | \bar{m} | k) \\ \gamma & 1 \end{smallmatrix}} A_{\begin{smallmatrix} (n | \bar{m} | k) \\ \alpha & 2 \\ (n | \bar{m} | k) \\ \gamma & 2 \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | \bar{m} | k) \\ \alpha & \rho \\ (n | \bar{m} | k) \\ \gamma & \rho \end{smallmatrix}} \right) \\ & = A^{(m-1)_{k-1}} A_{\begin{smallmatrix} (m-1)_k \\ (n | \bar{m}) \\ \alpha \\ (n | \bar{m}) \\ \gamma \end{smallmatrix}} \quad (\text{Art. 12}). \end{aligned}$$

* Cf. Scott, Proc. Lon. Math. Soc., vol. XIV, §8.

† Cf. Scott, ibid. art. 6, p. 96.

The product

$$\begin{aligned}
 & \left(A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & 1) \\ (n & \gamma & | & 1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & 2) \\ (n & \gamma & | & 2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & i) \\ (n & \gamma & | & i) \end{smallmatrix}} \right)_{i < \rho} \\
 & \quad \times \left(A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & 1) \\ (n & \gamma & | & 1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & 2) \\ (n & \gamma & | & 2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & \rho) \\ (n & \gamma & | & \rho) \end{smallmatrix}} \right) \\
 & = A^i A_{\begin{smallmatrix} (n & m & | & m) \\ (n & \alpha & | & i) \\ (n & \gamma & | & m) \end{smallmatrix}}^i \left(A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & i+1) \\ (n & \gamma & | & i+1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & i+2) \\ (n & \gamma & | & i+2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & \rho) \\ (n & \gamma & | & \rho) \end{smallmatrix}} \right), \\
 & \therefore \left(A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & 1) \\ (n & \gamma & | & 1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & 2) \\ (n & \gamma & | & 2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & m & | & k) \\ (n & \alpha & | & i) \\ (n & \gamma & | & i) \end{smallmatrix}} \right) \\
 & = A^{i-(m-1)k-1} A_{\begin{smallmatrix} (n & m & | & m) \\ (n & \alpha & | & i) \\ (n & \gamma & | & m) \end{smallmatrix}}^{i-(m-1)k} \left(A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & i+1) \\ (n & \gamma & | & i+1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & i+2) \\ (n & \gamma & | & i+2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & \bar{m} & | & k) \\ (n & \alpha & | & \rho) \\ (n & \gamma & | & \rho) \end{smallmatrix}} \right) \quad (17)
 \end{aligned}$$

If A has a block of zero constituents, its value may be given as in art. 14.*

17. The product

$$\begin{aligned}
 & \left(A_{\begin{smallmatrix} (n & m & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m & | & m-2) \\ (n & 1 & | & m-2) \\ (n & 1 & | & m-2) \end{smallmatrix}} A_{\begin{smallmatrix} (\bar{n} & n-1 & | & 1) \\ (\bar{n} & 1 & | & 1) \\ (\bar{n} & 1 & | & 1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m & | & m-3) \\ (n & 1 & | & m-3) \\ (n & 1 & | & m-3) \end{smallmatrix}} A_{\begin{smallmatrix} (\bar{n} & n-2 & | & 1) \\ (\bar{n} & 1 & | & 1) \\ (\bar{n} & 1 & | & 1) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (\bar{n} & n-m+1 & | & 1) \\ (\bar{n} & 1 & | & 1) \\ (\bar{n} & 1 & | & 1) \end{smallmatrix}} \right) \\
 & \quad \times \left(A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} \right) \\
 & = A^m A_{\begin{smallmatrix} (n & m & | & m) \\ (n & 1 & | & m) \\ (n & 1 & | & m) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m-1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m-2 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \end{smallmatrix}} A_{\begin{smallmatrix} (n & m-3 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & 1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \\ (n & 1 & | & \bar{n}) \end{smallmatrix}} A_{\begin{smallmatrix} (\bar{n} & n-m+1 & | & 1) \\ (\bar{n} & 1 & | & 1) \\ (\bar{n} & 1 & | & 1) \end{smallmatrix}}
 \end{aligned}$$

the product of the constituents along the principal diagonal of the product. The truth of this is seen on observing that all the constituents on the lower left-hand side of the principal diagonal of the product are zero.

The determinant

$$\left(A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n & \bar{m} & | & m-1) \\ (n & 1 & | & m-1) \\ (n & 1 & | & m-1) \end{smallmatrix}} \right),$$

* Cf. Scott, *ibid.* art. 9.

being a minor of order m of $A_{(n-1)}$ is equal to

$$A^{m-1} A_{\begin{smallmatrix} (n | m) \\ 1 \end{smallmatrix}}.$$

Therefore

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | m-1) \\ (n | m | m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | m-2) (\bar{n} | n-1) \\ (n | m | m-2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (\bar{n} | n-m+1) \\ (n | m | m-1) \end{smallmatrix}} \right) \\ &= A \cdot A_{\begin{smallmatrix} (n | m-1) (\bar{n} | n-1) \\ (n | m) \end{smallmatrix}} \cdot A_{\begin{smallmatrix} (n | m-2) (\bar{n} | n-2) \\ (n | m) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | 1) (\bar{n} | n-m+1) \\ (n | m) \end{smallmatrix}}. \end{aligned} \quad (18)$$

This is the theorem given by Muir in his "Theory of Determinants," §93.

18. The product

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | m-1) \\ (n | m | m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | m-2) (\bar{n} | n-1) \\ (n | m | m-2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | m-l) (\bar{n} | n-l+1) \\ (n | m | m-l) \end{smallmatrix}} \right) \\ & \quad \times \left(A_{\begin{smallmatrix} (n | \bar{m} | m-1) \\ (n | \bar{m} | m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n | \bar{m} | m-2) \\ (n | \bar{m} | m-2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | \bar{m} | m-1) \\ (n | \bar{m} | m-1) \end{smallmatrix}} \right) \\ &= A^l \cdot A_{\begin{smallmatrix} (n | m) \\ (n | m) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m-1) (\bar{n} | n-1) \\ (n | m) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m-l+1) (\bar{n} | n-l+1) \\ (n | m) \end{smallmatrix}} \\ & \quad \times \left(A_{\begin{smallmatrix} (n | \bar{m} | m-1) \\ (n | \bar{m} | m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n | \bar{m} | m-2) \\ (n | \bar{m} | m-2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | \bar{m} | m-1) \\ (n | \bar{m} | m-1) \end{smallmatrix}} \right) \\ &= A^l \cdot A_{\begin{smallmatrix} (n | m) \\ (n | m) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m-1) (\bar{n} | n-1) \\ (n | m) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | \bar{m} | m-1) \\ (n | \bar{m} | m-1) \end{smallmatrix}} \\ & \quad \times A^{m-l-1} \cdot A_{\begin{smallmatrix} (n | m-1) \\ (n | m-1) \end{smallmatrix}} \\ &= A^{m-1} A_{\begin{smallmatrix} (n | m) \\ (n | m) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m-1) (\bar{n} | n-1) \\ (n | m-1) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m-l+1) (\bar{n} | n-l+1) \\ (n | m) \end{smallmatrix}}. \end{aligned}$$

Therefore

$$\begin{aligned} & \left(A_{\begin{smallmatrix} (n | m | m-1) \\ (n | m | m-1) \end{smallmatrix}} A_{\begin{smallmatrix} (n | m | m-2) (\bar{n} | n-1) \\ (n | m | m-2) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m | m-l) (\bar{n} | n-l+1) \\ (n | m | m-l) \end{smallmatrix}} \right) \\ &= A_{\begin{smallmatrix} (n | m-1) (\bar{n} | n-1) \\ (n | m-1) \end{smallmatrix}} \cdots A_{\begin{smallmatrix} (n | m-l+1) (\bar{n} | n-l+1) \\ (n | m) \end{smallmatrix}}. \end{aligned} \quad (19)$$

Operating on equation (18) by the *Law of Complementaries* we get

$$\begin{aligned} & \left(A_{\overline{n} \mid m-1 \atop 1} A_{(n \mid \overline{n-1} \mid m-2) \atop 1} A_{(n \mid \overline{n-2} \mid m-3) \atop 1} \cdots A_{(n \mid \overline{n-m+1} \mid 1) \atop 1} \right) \\ &= A_{(n \mid \overline{n-1} \mid m-1) \atop (\overline{n} \mid m) \atop 1} A_{(n \mid \overline{n-2} \mid m-2) \atop (\overline{n} \mid m) \atop 1} \cdots A_{(n \mid \overline{n-m+1} \mid 1) \atop (\overline{n} \mid m) \atop 1}. \end{aligned} \quad (20)$$

19. The determinant obtained on writing $(n \mid m \mid l \mid k)_{\alpha \gamma \delta} (\overline{n} \mid \overline{m} \mid l \mid k)_{\alpha \gamma \epsilon}$ for $(n \mid m \mid l)_{\alpha \gamma}$ in

$$\left(A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop 1} A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop 2} \cdots A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop \lambda} \right)$$

evidently has two rows identical and therefore vanishes.

20. The determinant M_1 , obtained on writing $(n \mid m \mid l \mid k)_{\alpha \gamma \delta} (\overline{n} \mid \overline{m} \mid l \mid k)_{\alpha \gamma \epsilon}$ for $(n \mid m \mid l)_{\alpha \gamma}^*$ in

$$\left(A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop 1} A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop 2} \cdots A_{(n \mid m \mid l) \atop (\overline{n} \mid \overline{m} \mid l) \atop \beta \atop \lambda} \right)$$

vanishes, though two rows are not identical. For if we multiply M_1 by

$$\left(A_{(n \mid \overline{m} \mid l) \atop (\overline{n} \mid m \mid l) \atop \beta \atop 1} A_{(n \mid \overline{m} \mid l) \atop (\overline{n} \mid m \mid l) \atop \beta \atop 2} \cdots A_{(n \mid \overline{m} \mid l) \atop (\overline{n} \mid m \mid l) \atop \beta \atop \lambda} \right),$$

every constituent in the γ^{th} column of the product will be zero since the upper line of every suffix of M_1 contains some number in common with the combination $(n \mid \overline{m} \mid l)_{\alpha \gamma}$. The product therefore vanishes, and since the multiplier is in general different from zero, M_1 must vanish.

21. Any determinant M_2 of order λ , the upper line of every suffix of which contains at least one of some $(m - l)$ or fewer numbers and the lower lines of the suffixes are the combinations l at a time of some m of the n numbers, vanishes.

This theorem, which is perhaps a little more comprehensive than that of the last article, is proven in a similar way.

* We may, of course, have similar substitutions in the upper lines of the suffixes of the constituents in other rows at the same time.

Let the numbers in the combination $(n | \overline{m} | l)_{\alpha \gamma}$ be the $(m-l)$ numbers in question, and let the lower lines of the suffixes be the combinations $(n | m | l)_{\beta 1}$, $(n | m | l)_{\beta 2} \dots (n | m | l)_{\beta \lambda}$.

If we multiply M_2 by

$$\left(A_{(n | \overline{m} | l)_{\alpha 1}} A_{(n | \overline{m} | l)_{\alpha 2}} \dots A_{(n | \overline{m} | l)_{\alpha \lambda}} \right),$$

$$\begin{matrix} (n | \overline{m} | l)_{\beta 1} & (n | \overline{m} | l)_{\beta 2} & & (n | \overline{m} | l)_{\beta \lambda} \end{matrix}$$

every constituent in the γ^{th} column of the product will be zero, since the upper line of every constituent of M_2 contains some number in common with the combination $(n | \overline{m} | l)_{\alpha \gamma}$. Therefore the product and consequently M_2 vanishes.

22. Every minor of the determinant M_2 of the last article, which is of order $\geq \{(m-k)_i + 1\}$ and which contains any k numbers in at least $\{(m-k)_i + 1\}$ of the upper lines of the suffixes, vanishes.

Without loss of generality we may suppose that the upper lines of the suffixes of the constituents in the first $\{(m-k)_i + 1\}$ rows of M_2 contain the same k numbers.

Multiplying any minor containing these rows by the same multiplier as in the last article, it is readily seen that every constituent in the intersection of the first $\{(m-k)_i + 1\}$ rows and last $\{m_i - (m-k)_i\}$ columns of the product are zero. Therefore the product and consequently the minor vanishes.

23. Any minor of

$$\left(A_{(n | m | l)_{\alpha 1}} A_{(n | m | l)_{\alpha 2}} \dots A_{(n | m | l)_{\alpha \lambda}} \right)$$

$$\begin{matrix} (n | m | l)_{\beta 1} & (n | m | l)_{\beta 2} & & (n | m | l)_{\beta \lambda} \end{matrix}$$

can be expressed as a product of A and its minors whenever the complementary of the corresponding minor of

$$\left(A_{(n | \overline{m} | l)_{\alpha 1}} A_{(n | \overline{m} | l)_{\alpha 2}} \dots A_{(n | \overline{m} | l)_{\alpha \lambda}} \right)$$

$$\begin{matrix} (n | \overline{m} | l)_{\beta 1} & (n | \overline{m} | l)_{\beta 2} & & (n | \overline{m} | l)_{\beta \lambda} \end{matrix}$$

can be so expressed.

If we know the value of any minor of $A_{(m)}$, the *Law of Complementaries* gives us the value of the corresponding minor of $A_{(n-m)}$, and the theorem of art. 1

gives us the value of the complementary of the corresponding minor of $A_{(n-m)}$, which again operated on by the *Law of Complementaries* gives the value of the complementary of the original minor of $A_{(m)}$. To determine the value of all the minors of $A_{(m)}$ and $A_{(n-m)}$ all that is necessary therefore is to know the value of the minors of any $\left\{\frac{n_m}{2}\right\}^*$ orders of one of them.

24. Let us illustrate the foregoing principles by finding the value of minors of $A_{(3)}$, the second compound of the determinant A of order five.

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} \\ 12 & 13 & 23 \\ A_{13} & A_{13} & A_{13} \\ 12 & 13 & 23 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix} = \begin{vmatrix} A \cdot A_{123} & 0 & 0 \\ 0 & A \cdot A_{123} & 0 \\ A_{31} & -A_{21} & A_{11} \end{vmatrix} = A^2 A_{11} A_{123}^2.$$

$$\therefore \left(A_{12} A_{13} \right)_{12} = A \cdot A_{123}.$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} \\ 12 & 13 & 14 & 15 \\ A_{13} & A_{13} & A_{13} & A_{13} \\ 12 & 13 & 14 & 15 \\ A_{14} & A_{14} & A_{14} & A_{14} \\ 12 & 13 & 14 & 15 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} \\ -a_{32} & a_{33} & -a_{34} & a_{35} \\ a_{42} & -a_{43} & a_{44} & -a_{45} \\ -a_{52} & a_{53} & -a_{54} & a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 0 \\ 0 & 0 & A_{11} & 0 \\ -a_{25} & a_{35} & -a_{45} & a_{55} \end{vmatrix} = a_{55} A_{11}^3,$$

$$\therefore \left(A_{12} A_{13} A_{14} \right)_{12} = a_{55} A_{11}^2 \text{ or } A_{11}^2 A_{1234}.$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} \\ 12 & 13 & 23 \\ A_{13} & A_{13} & A_{13} \\ 12 & 13 & 23 \\ A_{23} & A_{23} & A_{23} \\ 12 & 13 & 23 \end{vmatrix} \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

* $\left\{\frac{n_m}{2}\right\}$ denotes the greatest integer in $\frac{n_m}{2}$.

$$= \begin{vmatrix} A \cdot A_{123} & 0 & 0 \\ 0 & A \cdot A_{123} & 0 \\ 0 & 0 & A \cdot A_{123} \end{vmatrix} = A^3 \cdot A_{123}^3,$$

$$\therefore \left(A_{12} \ A_{13} \ A_{23} \right) = A \cdot A_{123}^2 \quad (\text{Eq. (8)}).$$

The product (by columns)

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} \\ 12 & 15 & 45 \\ A_{13} & A_{13} & A_{13} \\ 12 & 15 & 45 \\ A_{23} & A_{23} & A_{23} \\ 12 & 15 & 45 \end{vmatrix} \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

$$= \begin{vmatrix} A \cdot A_{123} & 0 & 0 \\ -A \cdot A_{123} & A \cdot A_{123} & 0 \\ A \cdot A_{123} & A \cdot A_{123} & A \cdot A_{123} \end{vmatrix} = A^3 \cdot A_{123} \ A_{123} \ A_{123},$$

$$\therefore \left(A_{12} \ A_{13} \ A_{23} \right) = A \cdot A_{123} \cdot A_{123} \quad (\text{Eq. (18)}).$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} \\ 12 & 13 & 23 \\ A_{13} & A_{13} & A_{23} \\ 12 & 13 & 23 \\ A_{25} & A_{25} & A_{25} \\ 12 & 13 & 23 \end{vmatrix} \begin{vmatrix} A_{33} & -A_{32} & A_{31} \\ -A_{23} & A_{22} & -A_{21} \\ A_{13} & -A_{12} & A_{11} \end{vmatrix}$$

$$= \begin{vmatrix} A \cdot A_{123} & 0 & 0 \\ 0 & A \cdot A_{123} & 0 \\ -A \cdot A_{235} & 0 & A \cdot A_{125} \end{vmatrix} = A^3 \cdot A_{123}^2 \ A_{125},$$

$$\therefore \left(A_{12} \ A_{13} \ A_{25} \right) = A \cdot A_{123} \ A_{125}.$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} & A_{12} & A_{12} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ A_{13} & A_{13} & A_{13} & A_{13} & A_{13} & A_{13} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{24} & A_{24} & A_{24} & A_{24} & A_{24} & A_{24} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_{34} & -A_{34} & A_{34} & A_{34} & -A_{34} & A_{34} \\ 34 & 24 & 23 & 24 & 13 & 12 \\ -A_{24} & A_{24} & -A_{24} & -A_{24} & A_{24} & -A_{24} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{23} & -A_{23} & A_{23} & A_{23} & -A_{23} & A_{23} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{14} & -A_{14} & A_{14} & A_{14} & -A_{14} & A_{14} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ -A_{13} & A_{13} & -A_{13} & -A_{13} & A_{13} & -A_{13} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{12} & -A_{12} & A_{12} & A_{12} & -A_{12} & A_{12} \\ 34 & 24 & 23 & 14 & 13 & 12 \end{vmatrix} \\
 = \begin{vmatrix} A \cdot a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A \cdot a_{55} & 0 & 0 & 0 & 0 \\ A_{34} & -A_{24} & A_{23} & A_{14} & -A_{13} & A_{12} \\ 23 & 23 & 23 & 23 & 23 & 23 \\ A_{34} & -A_{24} & A_{23} & A_{14} & -A_{13} & A_{12} \\ 14 & 14 & 14 & 14 & 14 & 14 \\ 0 & 0 & 0 & 0 & A \cdot a_{55} & 0 \\ A_{34} & -A_{24} & A_{23} & A_{14} & -A_{13} & A_{12} \\ 12 & 12 & 12 & 12 & 12 & 12 \end{vmatrix} = A^3 \cdot a_{55}^3 \begin{vmatrix} A_{12} & A_{14} & A_{23} \\ 12 & 14 & 23 \end{vmatrix}, \\
 \therefore \begin{vmatrix} A_{12} & A_{13} & A_{24} \\ 12 & 13 & 24 \end{vmatrix} = \begin{vmatrix} A_{12} & A_{14} & A_{23} \\ 12 & 14 & 23 \end{vmatrix}.$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} \\ 12 & 13 & 23 \\ A_{13} & A_{13} & A_{13} \\ 12 & 13 & 23 \\ A_{14} & A_{14} & A_{14} \\ 12 & 13 & 23 \end{vmatrix} \begin{vmatrix} A_{33} & A_{32} & A_{31} \\ A_{23} & A_{22} & A_{21} \\ A_{13} & A_{12} & A_{11} \end{vmatrix} = \begin{vmatrix} A \cdot A_{123} & 0 & 0 \\ 123 & & \\ 0 & A \cdot A_{122} & 0 \\ & 123 & \\ -A \cdot A_{124} & A \cdot A_{124} & 0 \\ & 123 & 123 \end{vmatrix} = 0, \\
 \therefore \begin{vmatrix} A_{12} & A_{13} & A_{14} \\ 12 & 13 & 23 \end{vmatrix} = 0 \quad (\text{arts. 20, 21}).$$

There are $2 \times 10 \times 20 = 400$ vanishing minors of order three.

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} \\ 12 & 13 & 14 & 15 \\ A_{13} & A_{13} & A_{13} & A_{13} \\ 12 & 13 & 14 & 15 \\ A_{14} & A_{14} & A_{14} & A_{14} \\ 12 & 13 & 14 & 15 \\ A_{15} & A_{15} & A_{15} & A_{15} \\ 12 & 13 & 14 & 15 \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{11} & 0 & 0 \\ 0 & 0 & A_{11} & 0 \\ 0 & 0 & 0 & A_{11} \end{vmatrix} = A_{11}^4, \\
 \therefore \begin{vmatrix} A_{12} & A_{13} & A_{14} & A_{15} \\ 12 & 13 & 14 & 15 \end{vmatrix} = A_{11}^3 \quad (\text{Eq. (7)}).$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} & A_{12} & A_{12} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ A_{13} & A_{13} & A_{13} & A_{13} & A_{13} & A_{13} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ A_{14} & A_{14} & A_{14} & A_{14} & A_{14} & A_{14} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ A_{23} & A_{23} & A_{23} & A_{23} & A_{23} & A_{23} \\ 12 & 13 & 14 & 23 & 24 & 34 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_{34} & -A_{34} & A_{34} & A_{34} & -A_{34} & A_{34} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ -A_{24} & A_{24} & -A_{24} & -A_{24} & A_{24} & -A_{24} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{23} & -A_{23} & A_{23} & A_{23} & -A_{23} & A_{23} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{14} & -A_{14} & A_{14} & A_{14} & -A_{14} & A_{14} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ -A_{13} & A_{13} & -A_{13} & -A_{13} & A_{13} & -A_{13} \\ 34 & 24 & 23 & 14 & 13 & 12 \\ A_{12} & -A_{12} & A_{12} & A_{12} & -A_{12} & A_{12} \\ 34 & 24 & 23 & 14 & 13 & 12 \end{vmatrix}$$

$$\begin{vmatrix} A \cdot a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A \cdot a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & A \cdot a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & A \cdot a_{55} & 0 & 0 \\ -A_{34} & A_{24} & -A_{23} & -A_{14} & A_{13} & -A_{12} \\ 13 & 13 & 13 & 13 & 13 & 13 \\ A_{34} & -A_{24} & A_{23} & A_{14} & -A_{13} & A_{12} \\ 12 & 12 & 12 & 12 & 12 & 12 \end{vmatrix} = A^4 \cdot a_{55}^4 \begin{pmatrix} A_{12} & A_{13} \\ 12 & 13 \end{pmatrix}$$

$$= A^4 \cdot a_{55}^4 A_{11} A_{123},$$

$$\therefore \begin{pmatrix} A_{12} & A_{13} & A_{14} & A_{23} \\ 12 & 13 & 14 & 23 \end{pmatrix} = A \cdot A_{11} A_{123} A_{1234}.$$

The product

$$\begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} \\ 12 & 13 & 14 & 15 \\ A_{23} & A_{23} & A_{23} & A_{23} \\ 12 & 13 & 14 & 15 \\ A_{34} & A_{34} & A_{34} & A_{34} \\ 12 & 13 & 14 & 15 \\ A_{45} & A_{45} & A_{45} & A_{45} \\ 12 & 13 & 14 & 15 \end{vmatrix} \begin{vmatrix} a_{22} & -a_{23} & a_{24} & -a_{25} \\ -a_{32} & a_{33} & -a_{34} & a_{35} \\ a_{42} & -a_{43} & a_{44} & -a_{45} \\ -a_{52} & a_{53} & -a_{54} & a_{55} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{21} & 0 & 0 \\ 0 & 0 & A_{31} & 0 \\ 0 & 0 & 0 & A_{41} \end{vmatrix} = A_{11} A_{21} A_{31} A_{41},$$

$$\therefore \begin{pmatrix} A_{12} & A_{23} & A_{34} & A_{45} \\ 12 & 13 & 14 & 15 \end{pmatrix} = A_{21} A_{31} A_{41} \quad (\text{Eq. (20)}).$$

The product

$$\begin{vmatrix}
 A_{12} & A_{12} & A_{12} & A_{12} & A_{12} & A_{12} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{13} & A_{13} & A_{13} & A_{13} & A_{13} & A_{13} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{14} & A_{14} & A_{14} & A_{14} & A_{14} & A_{14} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{15} & A_{15} & A_{15} & A_{15} & A_{15} & A_{15} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{vmatrix}
 \begin{vmatrix}
 A_{34} & -A_{34} & A_{34} & A_{34} & -A_{34} & A_{34} \\
 34 & 24 & 23 & 14 & 13 & 12 \\
 -A_{24} & A_{24} & -A_{24} & -A_{24} & A_{24} & -A_{24} \\
 34 & 24 & 23 & 14 & 13 & 12 \\
 A_{23} & -A_{23} & A_{23} & A_{23} & -A_{23} & A_{23} \\
 34 & 24 & 23 & 14 & 13 & 12 \\
 A_{14} & -A_{14} & A_{14} & A_{14} & -A_{14} & A_{14} \\
 34 & 24 & 23 & 14 & 13 & 12 \\
 -A_{13} & A_{13} & -A_{13} & -A_{13} & A_{13} & -A_{13} \\
 34 & 24 & 23 & 14 & 13 & 12 \\
 A_{12} & -A_{12} & A_{12} & A_{12} & -A_{12} & A_{12} \\
 34 & 24 & 23 & 14 & 13 & 12
 \end{vmatrix}$$

$$= \begin{vmatrix}
 A \cdot a_{55} & 0 & 0 & 0 & 0 & 0 \\
 0 & A \cdot a_{55} & 0 & 0 & 0 & 0 \\
 0 & 0 & A \cdot a_{55} & 0 & 0 & 0 \\
 A \cdot a_{25} & A \cdot a_{35} & A \cdot a_{45} & 0 & 0 & 0 \\
 -A_{34} & A_{24} & -A_{23} & -A_{14} & A_{13} & -A_{12} \\
 13 & 13 & 13 & 13 & 13 & 13 \\
 A_{34} & -A_{24} & A_{23} & A_{14} & -A_{13} & A_{12} \\
 12 & 12 & 12 & 12 & 12 & 12
 \end{vmatrix} = 0.$$

$$\therefore (A_{12} A_{13} A_{14} A_{15}) = 0 \quad (\text{art. 22}).$$

From this it is apparent that any determinant of order four formed from the matrix

$$\begin{vmatrix}
 A_{12} & A_{12} & A_{12} & A_{12} & A_{12} & A_{12} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{13} & A_{13} & A_{13} & A_{13} & A_{13} & A_{13} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{14} & A_{14} & A_{14} & A_{14} & A_{14} & A_{14} \\
 12 & 13 & 14 & 23 & 24 & 34 \\
 A_{15} & A_{15} & A_{15} & A_{15} & A_{15} & A_{15} \\
 12 & 13 & 14 & 23 & 24 & 34
 \end{vmatrix}$$

vanishes.

There are $2 \cdot \{75 \times 5\} = 750$ vanishing minors of order four.

The product

$$\begin{vmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 A_{24} & A_{24} & A_{24} & A_{24} & A_{24} & A_{24} \\
 23 & 24 & 25 & 34 & 35 & 45 \\
 A_{25} & A_{25} & A_{25} & A_{25} & A_{25} & A_{25} \\
 23 & 24 & 25 & 34 & 35 & 45 \\
 A_{34} & A_{34} & A_{34} & A_{34} & A_{34} & A_{34} \\
 23 & 24 & 25 & 34 & 35 & 45 \\
 A_{35} & A_{35} & A_{35} & A_{35} & A_{35} & A_{35} \\
 23 & 24 & 25 & 34 & 35 & 45 \\
 A_{45} & A_{45} & A_{45} & A_{45} & A_{45} & A_{45} \\
 23 & 24 & 25 & 34 & 35 & 45
 \end{vmatrix}
 \begin{vmatrix}
 A_{45} & -A_{45} & A_{45} & A_{45} & -A_{45} & A_{45} \\
 45 & 35 & 34 & 25 & 24 & 23 \\
 -A_{35} & A_{35} & -A_{35} & -A_{35} & A_{35} & -A_{35} \\
 45 & 35 & 34 & 25 & 24 & 23 \\
 A_{34} & -A_{34} & A_{34} & A_{34} & -A_{34} & A_{34} \\
 45 & 35 & 34 & 25 & 24 & 23 \\
 A_{25} & -A_{25} & A_{25} & A_{25} & -A_{25} & A_{25} \\
 45 & 35 & 34 & 25 & 24 & 23 \\
 -A_{24} & A_{24} & -A_{24} & -A_{24} & A_{24} & -A_{24} \\
 45 & 35 & 34 & 25 & 24 & 23 \\
 A_{23} & -A_{23} & A_{23} & A_{23} & -A_{23} & A_{23} \\
 45 & 35 & 34 & 25 & 24 & 23
 \end{vmatrix}$$

$$= \begin{vmatrix} A_{45} - A_{35} & A_{34} - A_{25} & A_{24} - A_{23} \\ 0 & A \cdot a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & A \cdot a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & A \cdot a_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & A \cdot a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & A \cdot a_{11} \end{vmatrix} = a_{11}^5 A^5 A_{45},$$

$$\therefore (A_{24} A_{25} A_{34} A_{35} A_{45}) = A^2 a_{11}^2 A_{45} \text{ or } A^2 A_{2345} A_{45}.$$

The product

$$= \begin{vmatrix} A_{12} & A_{12} & A_{12} & A_{12} & A_{12} & A_{12} \\ A_{13} & A_{13} & A_{13} & A_{13} & A_{13} & A_{13} \\ A_{14} & A_{14} & A_{14} & A_{14} & A_{14} & A_{14} \\ A_{15} & A_{15} & A_{15} & A_{15} & A_{15} & A_{15} \\ A_{23} & A_{23} & A_{23} & A_{23} & A_{23} & A_{23} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_{34} - A_{24} & A_{34} - A_{24} & A_{34} - A_{24} & A_{34} - A_{24} & A_{34} - A_{24} & A_{34} - A_{24} \\ -A_{24} & -A_{24} & -A_{24} & -A_{24} & -A_{24} & -A_{24} \\ A_{23} - A_{13} & A_{23} - A_{13} & A_{23} - A_{13} & A_{23} - A_{13} & A_{23} - A_{13} & A_{23} - A_{13} \\ A_{14} - A_{13} & A_{14} - A_{13} & A_{14} - A_{13} & A_{14} - A_{13} & A_{14} - A_{13} & A_{14} - A_{13} \\ -A_{13} & -A_{13} & -A_{13} & -A_{13} & -A_{13} & -A_{13} \\ A_{12} - A_{24} & A_{12} - A_{24} & A_{12} - A_{24} & A_{12} - A_{24} & A_{12} - A_{24} & A_{12} - A_{24} \end{vmatrix}$$

$$= \begin{vmatrix} A \cdot a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & A \cdot a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & A \cdot a_{55} & 0 & 0 & 0 \\ A \cdot a_{25} & A \cdot a_{35} & A \cdot a_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & A \cdot a_{55} & 0 & 0 \\ A_{34} - A_{24} & A_{23} & A_{14} - A_{13} & A_{12} \end{vmatrix} = 0,$$

$$\therefore (A_{12} A_{13} A_{14} A_{15} A_{23}) = 0.$$

There are $\{6.5\}^2 = \{30\}^2$ vanishing minors of order five.

SYRACUSE UNIVERSITY, Sept. 1, 1897.